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Characteristic initial value problem for the development of a Kerr solution

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Abstract. Some conditions on the characteristic initial data necessary for a solution of Einstein's equations given by a Bondi–Sachs metric to develop into a Kerr solution are determined. A special case is exhibited where some data specified on the initial null hypersurface, together with the news functions, determines the parameter values for the resulting Kerr solution.

1. Introduction

It is widely conjectured (see for example Hawking and Ellis 1973) that uncharged stellar bodies of mass greater than twice the mass of the sun will eventually collapse to a black hole which is described by a Kerr solution. In this paper some algebraic conditions necessary for this to occur are investigated by studying the characteristic initial value problem for an isolated body in otherwise empty, asymptotically flat, space–time (Bondi *et al* 1962, Sachs 1962, van der Burg 1966). It is shown that for a particular Kerr solution to develop, as well as the requirement that the news functions vanish after a certain time parameter value (which is equivalent to requiring the body to eventually stop radiating) the data which, in the normal initial value problem, could be specified arbitrarily, are now determined by the news functions and the parameters describing the Kerr solution.

In a special case where the news functions take a particular form, it is possible to specify two constants on an initial null hypersurface, which, together with the news functions, determine the final values of the parameters in the Kerr solution.

The analysis of the metric and field equations is based on the work of Bondi *et al* (1962) and Sachs (1962), but the version quoted here follows van der Burg (1966) which is briefly summarized in § 2. In § 3 the Boyer–Lindquist form for the Kerr metric (Boyer and Lindquist 1967) is transformed into the Bondi–Sachs coordinate system and the final values for the data variables obtained. These are imposed on the initial data in § 4 and the results of this procedure analysed. Some conclusions are summarized in § 5.

2. The characteristic initial value problem

The metric used by Sachs (1962) to describe asymptotically flat space–time, far away from a bounded source, is, in the form given by van der Burg (1966):

$$\begin{aligned} ds^2 = & (Vr^{-1}e^{2\beta} - r^2e^{2\gamma}U^2 \cosh 2\delta - r^2e^{-2\gamma}W^2 \cosh 2\delta - 2r^2UW \sinh 2\delta)\, du^2 \\ & + 2e^{2\beta}\, du\, dr + 2r^2(e^{2\gamma}U \cosh 2\delta + W \sinh 2\delta)\, du\, d\theta \\ & + 2r^2(e^{-2\gamma}W \cosh 2\delta + U \sinh 2\delta) \sin\theta\, du\, d\phi \\ & - r^2(e^{2\gamma} \cosh 2\delta\, d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2\theta\, d\phi^2 \\ & + 2 \sinh 2\delta \sin\theta\, du\, d\phi) \end{aligned} \tag{2.1}$$

where the coordinate system is based on a family of null hypersurfaces parameterized by $x^0 = u$; $x^1 = r$ is the luminosity distance along the null geodesics generating the hypersurfaces and $x^2 = \theta$ and $x^3 = \phi$ label the null geodesics. The functions β, γ, δ, U, V and W depend on these coordinates.

The form of the field equations and the method of solution are given in van der Burg (1966). In order to obtain explicit solutions, γ and δ are expanded in inverse powers of r in the following way,

$$\gamma = c(u, \theta, \phi)r^{-1} + (C(u, \theta, \phi) - \tfrac{1}{6}c^3 - \tfrac{3}{2}cd^2)r^{-3} + \ldots \tag{2.2}$$

$$\delta = d(u, \theta, \phi)r^{-1} + (H(u, \theta, \phi) + \tfrac{1}{2}c^2d - \tfrac{1}{6}d^3)r^{-3} + \ldots \tag{2.3}$$

where c, d, C and H are arbitrary functions of their arguments. Four of the field equations now enable the leading terms of β, U, V and W to be obtained in terms of c, d, C and H, together with three non-zero constants of r integration, $N(u, \theta, \phi)$, $P(u, \theta, \phi)$ and $M(u, \theta, \phi)$. The r^{-2} terms in (2.2) and (2.3) are omitted to prevent U and W having logarithmic terms, which would violate the outgoing radiation condition (Bondi *et al* 1962). Five other field equations determine the u derivatives of M, N, P, C and H, leaving one field equation which is now trivially satisfied. With derivatives being denoted by the appropriate subscript these five equations are:

$$\begin{aligned} M_0 = & -(c_0^2 + d_0^2) + \tfrac{1}{2}[(c_2 + 2c \cot\theta + d_3 \operatorname{cosec}\theta)_2 + (c_2 + 2c \cot\theta + d_3 \operatorname{cosec}\theta) \cot\theta \\ & + (d_2 + 2d \cot\theta - c_3 \operatorname{cosec}\theta)_3 \operatorname{cosec}\theta]_0 \end{aligned} \tag{2.4}$$

$$\begin{aligned} 3N_0 = & -M_2 - \tfrac{1}{2}\lambda_3 \operatorname{cosec}\theta - (c_0c_2 + d_0d_2) - 3(cc_{02} + dd_{02}) - 4(cc_0 + dd_0) \cot\theta \\ & + (c_0d_3 - c_3d_0 + 3c_{03}d - 3cd_{03}) \operatorname{cosec}\theta \end{aligned} \tag{2.5}$$

$$\begin{aligned} 3P_0 = & -M_3 \operatorname{cosec}\theta + \tfrac{1}{2}\lambda_2 + (c_2d_0 - c_0d_2) + 3(cd_{02} - c_{02}d) + 4(cd_0 - c_0d) \cot\theta \\ & - (c_0c_3 + d_0d_3 + 3cc_{03} + 3dd_{03}) \operatorname{cosec}\theta \end{aligned} \tag{2.6}$$

$$C_0 = \tfrac{1}{2}c^2c_0 + cdd_0 - \tfrac{1}{2}c_0d^2 + \tfrac{1}{2}cM + \tfrac{1}{4}d\lambda - \tfrac{1}{4}(N_2 - N \cot\theta - P_3 \operatorname{cosec}\theta) \tag{2.7}$$

$$H_0 = cc_0d - \tfrac{1}{2}c^2d_0 + \tfrac{1}{2}d^2d_0 + \tfrac{1}{2}dM - \tfrac{1}{4}c\lambda - \tfrac{1}{4}(P_2 - P \cot\theta + N_3 \operatorname{cosec}\theta) \tag{2.8}$$

where

$$\lambda = \left(\frac{\partial}{\partial\theta} + \cot\theta\right)(d_2 + 2d \cot\theta - c_3 \operatorname{cosec}\theta) - \operatorname{cosec}\theta \frac{\partial}{\partial\phi}(c_2 + 2c \cot\theta + d_3 \operatorname{cosec}\theta).$$

Thus the field equations determine the u derivatives of all the functions in the metric except those of c and d which are the arbitrary 'news functions' for the system.

A solution to the characteristic initial value problem is given by specifying C, H, N, P and M on a null hypersurface and giving c and d as functions of u, θ and ϕ. This treatment can be extended to cover as many coefficients in the expansions of γ and δ as is required. The initial values of these coefficients must be specified and then their subsequent development is determined.

3. The Kerr metric

The Boyer–Lindquist (1967) form of the metric for the Kerr solution is

$$\mathrm{d}s^2 = (1-2m\bar{r}\rho)\,\mathrm{d}\bar{u}^2 + 2\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{r} + 4m\bar{r}\rho a \sin^2\bar{\theta}\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{\phi} - 2a\sin^2\bar{\theta}\,\mathrm{d}\bar{r}\,\mathrm{d}\bar{\phi}$$
$$-\rho^{-1}\,\mathrm{d}\bar{\theta}^2 - [(\bar{r}^2+a^2)\sin^2\bar{\theta} + 2m\bar{r}\rho a^2\sin^4\bar{\theta}]\,\mathrm{d}\bar{\phi}^2 \qquad (3.1)$$

where $\rho = (\bar{r}^2 + a^2\cos^2\bar{\theta})^{-1}$. The coordinates are $\bar{u}, \bar{r}, \bar{\theta}$ and $\bar{\phi}$, which are not the coordinates introduced in the previous section, m represents the constant mass of the source and ma its constant angular momentum. This metric will now be transformed into the Bondi–Sachs form of metric (2.1) with the additional simplifications due to the Kerr metric being axially symmetric and non-radiative. These are, respectively, that β, γ, δ, U, V and W are independent of ϕ and that the news functions c_0 and d_0 vanish.

As the metric variables in the Bondi–Sachs metric are expanded in inverse powers of r, the transformation adopted here will be of the form

$$\bar{u} = \overset{-1}{b}\,r + \overset{0}{b} + r^{-1}\overset{1}{b} + \ldots$$
$$\bar{r} = Kr + \overset{0}{\rho} + r^{-1}\overset{1}{\rho} + \ldots$$
$$\bar{\theta} = \overset{-1}{g}\,r + \overset{0}{g} + r^{-1}\overset{1}{g} + \ldots$$
$$\bar{\phi} = \overset{-1}{h}\,r + \overset{0}{h} + r^{-1}\overset{1}{h} + \ldots \qquad (3.2)$$

where all the coefficients are functions of u, θ and ϕ. The notation employed here is reminiscent of the BMS transformations of the Bondi–Sachs metric (Bondi *et al* 1962), but the aim here is to transform the metric (3.1) into the form (2.1).

The mechanism of carrying out the transformation is tedious but not difficult, and by calculating each transformed Kerr metric component and identifying it with the corresponding component of the axi-symmetric Bondi–Sachs metric with zero news functions, results in the transformation being given by

$$\bar{u} = u + \tfrac{1}{2}r^{-1}a^2\sin^2\theta + \tfrac{1}{2}r^{-3}a^4\sin^2\theta(5\sin^2\theta/4 - 1) + \ldots$$
$$\bar{r} = r - \tfrac{1}{2}r^{-1}a^2\sin^2\theta - \tfrac{1}{2}r^{-2}ma^2\sin^2\theta + \ldots$$
$$\bar{\theta} = \theta - \tfrac{1}{2}r^{-2}a^2\sin\theta\cos\theta + r^{-4}\overset{4}{g}(\theta,\phi) + \ldots$$
$$\bar{\phi} = \phi + r^{-1}a + r^{-3}a^3(\tfrac{1}{2}\sin^2\theta - \tfrac{1}{3}) + r^{-4}\overset{4}{h}(\theta,\phi) + \ldots \qquad (3.3)$$

and the metric coefficients being

$$c = d = N = H = 0,\ M = m,\ C = -\tfrac{1}{2}ma^2\sin^2\theta,\ P = ma\sin\theta. \qquad (3.4)$$

4. Constraints on the initial data

In this section the constraints on the initial data for a solution given by the Bondi–Sachs metric to develop into a Kerr solution are discussed.

Suppose the parameters specifying a particular Kerr solution, m and a, are known. A first requirement to be satisfied is that after a certain u parameter value, say u_F, the source ceases to radiate, that is, c_0 and d_0 vanish for all $u \geqslant u_F$. Suppose further that, subject to this requirement, c_0 and d_0 are given as some functions of u, θ and ϕ. Now consider the propagation equation for M, (2.4), and integrate this from a value of u just less than u_F, say u_{F-1}, to u_F, where $M = m$. This gives the value for M on the u_{F-1} hypersurface. By repeating this integration process from hypersurface to hypersurface, the value of M on each hypersurface from u_F to u_I is obtained, and in particular the initial value of M is determined.

Next take the P propagation equation (2.6) and apply the same procedure, taking the final value of P to be $ma \sin\theta$ (from 3.4) and using the values of M previously obtained to approximate the term $\int_{u_i}^{u_{i+1}} -M_3 \operatorname{cosec}\theta \,.\, du$, where u_i and u_{i+1} parameterize neighbouring hypersurfaces. Hence the value of P is obtained initially and on all hypersurfaces between u_I and u_F.

In a similar manner the initial values of N, C and H can be calculated from equations (2.5), (2.7) and (2.8) and so the required constraints are that c_0 and d_0 vanish for $u \geqslant u_F$ and that the initial values of the data on the u_I hypersurface must take the values just obtained.

An alternative approach to this problem is to try to specify some data on the u_I hypersurface, which together with c_0 and d_0, determines the m and a parameters in the eventual Kerr solution. However there seems no way to achieve this in general, as both c_0 and d_0 have arbitrary θ and ϕ dependence which would prevent the final values for m and a being constant and which would lead to the initial values of the data having angular singularities. In a special case where $d = 0$ and $c = b\sin^2\theta$, where b is an arbitrary function of u, this objective can be achieved to the extent that the otherwise determined initial values of M and P each have one parameter available to be specified. The initial values of C, H and N are again completely determined.

In this case (2.4) yields, on integration from u_I to u_F,

$$m = M_I + \int_{u_I}^{u_F} (-)a_0^2 \, du \sin^4\theta + 6a_I \sin^2\theta - 4a_I$$

where a_I is the initial value of a, so if M_I is given by

$$M_I = -\int_{u_I}^{u_F} a_0^2 \, du \sin^4\theta + 6a_I \sin^2\theta + k$$

where k is a constant available to be specified, then $m = k - 4a_I$. For a physically realistic system $a_I < k/4$ and $k > 0$.

Equation (2.6) results in $P_0 = 0$, so if the initial value of P is given by $P_I = l\sin\theta$, with l a constant to be specified, then $a = l/m$. The other propagation equations determine the initial values of C, H and N, which with the choice of data given here, do not possess singularities at $\theta = 0, \pi$.

5. Conclusion

The constraints on the initial data for a solution corresponding to a Bondi–Sachs metric to develop into a particular Kerr solution are found and result in the initial values of C, H, N, M and P being determined. In addition the news functions must vanish after a given u parameter value. A special case is exhibited where some data can be given on the initial hypersurface, which together with a particular form for the news functions determines the parameter values for the eventual Kerr solution.

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